Stepwise structure of Lyapunov spectra for many-particle systems using a random matrix dynamics

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The structure of the Lyapunov spectra for the many-particle systems with a random interaction between the particles is discussed. The dynamics of the tangent space is expressed as a master equation, which leads to a formula that connects the positive Lyapunov exponents and the time correlations of the particle interaction matrix. Applying this formula to one- and two-dimensional models we investigate the stepwise structure of the Lyapunov spectra that appear in the region of small positive Lyapunov exponents. Long range interactions lead to a clear separation of the Lyapunov spectra into a part exhibiting stepwise structure and a part changing smoothly. The part of the Lyapunov spectrum containing the stepwise structure is clearly distinguished by a wavelike structure in the eigenstates of the particle interaction matrix. The two-dimensional model has the same step widths as found numerically in a deterministic chaotic system of many hard disks.

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I. INTRODUCTION

Chaos is characterized by a rapid expansion of a small initial error, and the Lyapunov exponent, which is defined as the time averaged exponential rate of the time evolution of infinitesimal perturbations of the dynamical variables, is introduced to express such characteristics quantitatively. The Lyapunov exponent is used to discuss some statistical properties of many-body systems (e.g., the mixing property). It is also known that the Lyapunov exponents are connected to the amount of information of the system (e.g., the Kolmogorov-Sinai entropy), and to the transport coefficients (e.g., conductance and viscosity) [1–3].

In general, the Lyapunov exponent depends on the direction of the infinitesimal perturbation of the dynamical variables at an initial time, so we obtain a Lyapunov exponent for each independent direction in the system. The sorted set of such Lyapunov exponents is the so called Lyapunov spectrum, and has been the subject of study in many chaotic particle systems. For example, the thermodynamic limit of the Lyapunov spectrum, that is, that the spectrum retains its shape as the number of particles increase is discussed using numerical evidence [4,5], random matrix approaches [6–8], a periodic orbit approach [9], and mathematical arguments [10,11]. Some works showed a linear behavior of the Lyapunov spectrum in one-dimensional models with nearest-neighbor interactions [4–8], although this conjecture is modified in weak chaotic systems [12,13]. The effect of the rotational degrees of freedom of molecules on the Lyapunov spectrum was investigated in a model consisting of diatomic molecules, which showed an explicit separation of the rotational and translational degrees of freedom if the departure from sphericity is small enough [14,15]. The Lyapunov spectra of systems in nonequilibrium steady states were investigated in Refs. [16–19] with the discovery of the conjugate pairing rule for some systems with isokinetic thermostats [20–23]. Avoided crossings and level repulsion in the Lyapunov spectrum, similar to the behavior of the energy levels in a quantum chaotic system, was discussed in a mutually coupled map system [24]. These works suggest the possibility of getting information about the system from the structure of its Lyapunov spectrum. The Lyapunov spectrum has also been of interest for the number dependence of the maximum Lyapunov exponent in particle systems [6,19,25–27], and the Kaplan-York Lyapunov dimension [28–31].

One of the characteristics of the Lyapunov spectrum, which has been shown recently in systems consisting of many hard disks, is its stepwise structure [14,15,19,32,33]. Such a stepwise structure appears in the distinct region of small positive Lyapunov exponents. Corresponding to this stepwise structure are the so called Lyapunov modes, a wavelike structure in the tangent space of each eigenvector of a degenerate Lyapunov exponent, that is, for each stepwise structure. An explanation of the stepwise structure of the Lyapunov spectrum due to symmetries of the system was tried using two-dimensional periodic orbit models [9]. On the other hand, explanations for the Lyapunov modes were suggested using a random matrix approach for a one-dimensional model [34] and more recently using a kinetic theoretical approach [35]. The small Lyapunov exponents correspond to slow relaxation processes in the equilibrium state, so these results suggest the possibility of characterizing the macroscopic behavior in many-particle systems from a microscopic point of view.

This paper has two main purposes. First we consider the dynamics of the tangent space in dynamical systems with a random interaction between the particles. Different from the other random matrix approaches for the Lyapunov spectrum [6,8,34,36], which use discretized models in time, we consider the continuous dynamics in time and derive a method to calculate the Lyapunov spectrum from a master equation technique. Using this method the Lyapunov spectrum is directly connected to the correlations of the particle interaction matrix. As the second purpose we discuss the stepwise structure of the Lyapunov spectrum using this random matrix approach. We consider one- and two-dimensional models satisfying the total momentum conservation with periodic boundary condition. These models show a stepwise structure of the Lyapunov spectra only in a region of small positive

Lyapunov exponents. We discuss the robustness of the stepwise structure against perturbations in the matrix elements of the particle interaction matrix. The long range interactions between the particles lead to a clear separation into a part of the spectrum exhibiting stepwise structure and a part changing smoothly with exponent number. We also find a wavelike structure in the eigenstates of the particle interaction matrix in the part of the Lyapunov spectrum containing stepwise structure. As a high dimensional effect we show that in the two-dimensional model we get wider steps in the Lyapunov spectrum, such as steps consisting of four and eight degenerate exponents, rather than simply two exponents as in the one-dimensional models. The step widths of four and eight exponents are actually observed in the numerical simulation of the many-hard-core-disk system [32].

II. RANDOM MATRIX DYNAMICS OF THE TANGENT VECTOR

We consider a Hamiltonian system whose state at time t is described by the $2N$ dimensional phase space vector $\mathbf{\Gamma}(t) \equiv (\mathbf{q}(t),\mathbf{p}(t))^T$ with the spatial coordinate vector $\mathbf{q}(t)$, the momentum vector $\mathbf{p}(t)$, and the transpose operation T. The dynamics of the phase space is described by the Hamiltonian equation

$$\frac{d\mathbf{\Gamma}(t)}{dt} = J\frac{\partial H(\mathbf{\Gamma}(t),t)}{\partial \mathbf{\Gamma}(t)}, \tag{1}$$

with the Hamiltonian $H(\mathbf{\Gamma}(t),t)$. Here we allow an explicit time dependence of the Hamiltonian, and J is the $2N\times 2N$ matrix defined by

$$J \equiv \begin{pmatrix} \underset{\sim}{0} & I \\ -I & \underset{\sim}{0} \end{pmatrix}, \tag{2}$$

where I is the $N\times N$ identical matrix and $\underset{\sim}{0}$ is the $N\times N$ null matrix. It should be noted that the matrix J satisfies the conditions $J^T = -J$ and $J^2 = -\overline{I}$ with the $2N\times 2N$ identical matrix $\overline{I}$.

The dynamics of the tangent vector $\delta\mathbf{\Gamma}(t)$ as a small deviation of the phase space vector is described by the linearized equation

$$\frac{d\,\delta\mathbf{\Gamma}(t)}{dt} = JL(t)\,\delta\mathbf{\Gamma}(t) \tag{3}$$

of the Hamiltonian equation (1) where $L(t)$ is defined by

$$L(t) \equiv \frac{\partial^2 H(\mathbf{\Gamma}(t),t)}{\partial\mathbf{\Gamma}(t)\partial\mathbf{\Gamma}(t)}. \tag{4}$$

The matrix $L(t)$ is symmetric: $L(t) = L(t)^T$. At this point we note that an equation equivalent to Eq. (3) can also be obtained for a system whose dynamics is not derivable from a Hamiltonian.

Equation (3) is formally solved as

$$\delta\mathbf{\Gamma}(t) = M(t,t_0)\,\delta\mathbf{\Gamma}(t_0), \tag{5}$$

where the matrix $M(t,t_0)$ is defined by

$$M(t,t_0) \equiv \overleftarrow{T}\exp\left[J\int_{t_0}^{t} ds\; L(s)\right], \tag{6}$$

with the positive time-ordering operator $\overleftarrow{T}$ with the latest time to the left. Equation (6) leads to

$$M(t,t_0)^{-1} = \overrightarrow{T}\exp\left[-J\int_{t_0}^{t} ds L(s)\right] = -JM(t,t_0)^T J, \tag{7}$$

with the negative time-ordering operator $\overrightarrow{T}$ with the latest time to the right, so that the Hamiltonian phase volume is preserved, namely, $|\mathrm{Det}\{M(t,t_0)\}| = 1$.

For simplicity, we consider the case of no external force field, so the Hamiltonian $H(\mathbf{\Gamma}(t),t)$ is represented as $H(\mathbf{\Gamma}(t),t) = |\mathbf{p}(t)|^2/(2m) + V(\mathbf{q}(t),t)$, where m is the mass of the particle and $V(\mathbf{q}(t),t)$ is the potential energy at time t. In this case the matrix $L(t)$ is given by

$$L(t) = \begin{pmatrix} -R(t) & \underset{\sim}{0} \\ \underset{\sim}{0} & I/m \end{pmatrix}, \tag{8}$$

where $R(t) \equiv (R_{\alpha\beta}(t))$ is a $N\times N$ symmetric matrix defined by $-\partial^2 V(\mathbf{q}(t),t)/\partial\mathbf{q}(t)\partial\mathbf{q}(t)$. The effect of particle interactions is taken into account through the matrix $R(t)$.

Now we move from considering particular Hamiltonian model systems to introducing a random interaction between the particles. We consider the case where the particle interactions occur randomly enough, so that the matrix elements $R_{\alpha\beta}(t)$, $\alpha = 1,2,\ldots,N$, $\beta = 1,2,\ldots,N$ can be regarded as a Gaussian white randomness in the sense of

$$\langle R_{\alpha_1\beta_1}(t_1)R_{\alpha_2\beta_2}(t_2)\cdots R_{\alpha_{2n-1}\beta_{2n-1}}(t_{2n-1})\rangle = 0, \tag{9}$$

$$\begin{aligned}\langle R_{\alpha_1\beta_1}(t_1)R_{\alpha_2\beta_2}(t_2)\cdots R_{\alpha_{2n}\beta_{2n}}(t_{2n})\rangle \\ = \sum_{P_d} D_{\alpha_{j_1}\beta_{j_1}\alpha_{j_2}\beta_{j_2}} D_{\alpha_{j_3}\beta_{j_3}\alpha_{j_4}\beta_{j_4}}\cdots D_{\alpha_{j_{2n-1}}\beta_{j_{2n-1}}\alpha_{j_{2n}}\beta_{j_{2n}}} \\ \times\delta(t_{j_1}-t_{j_2})\,\delta(t_{j_3}-t_{j_4})\cdots\delta(t_{j_{2n-1}}-t_{j_{2n}})\end{aligned} \tag{10}$$

for any integer n, where we take the sum over only the permutation P_d: $(1,2,\ldots,2n)\to(j_1,j_2,\ldots,j_{2n})$, and the bracket $\langle\cdots\rangle$ means the ensemble average over random processes. Here D_{jkln} is a fourth rank tensor and is assumed to be constant in the rest of this paper.

In order to justify the δ function relaxation (10) of the time correlations as a description of a deterministic chaotic system whose characteristic correlation time scale is not infinitesimal, it may be necessary to modify the time scale by a finite correlation time. Such a change of the time scale multiplies the Lyapunov exponents by a factor. However, in this paper we will only consider the ratios of the Lyapunov exponents, especially the Lyapunov exponents divided by the largest Lyapunov exponent, and for these quantities the problem of the time scale no longer appears explicitly. It may also be noted that the random particle interactions expressed by Eqs. (9) and (10) can be derived formally from a harmonic oscillator interaction potential with time-dependent Gaussian white-noise amplitudes.

The tensor D_{jkln} satisfies the condition

$$D_{lnjk}=D_{jkln}, \tag{11}$$

because the equation $D_{jkln}\delta(s-t)=\langle R_{jk}(s)R_{ln}(t)\rangle=\langle R_{ln}(t)R_{jk}(s)\rangle=D_{lnjk}\delta(s-t)$ is satisfied at arbitrary times s and t. The symmetric property of the matrix $R(t)$ also imposes the conditions

$$D_{jknl}=D_{kjln}=D_{jkln} \tag{12}$$

for the tensor D_{jkln}.

Under the Gaussian white conditions (9) and (10) for the matrix $R(t)$, the dynamics described by Eq. (3) for the tangent vector can be regarded as a stochastic process. Now we consider the description of this stochastic process by a master equation for the probability density $\rho(\delta\mathbf{\Gamma},t)$ for the tangent vector space in which $\rho(\delta\mathbf{\Gamma},t)d\delta\mathbf{\Gamma}$ is introduced as the probability of finding a tangent vector $\delta\mathbf{\Gamma}$ in the region $(\delta\mathbf{\Gamma},\delta\mathbf{\Gamma}+d\delta\mathbf{\Gamma})$. By applying the Kramers-Moyal expansion to the dynamics (3) with the randomness given by Eqs. (9) and (10) we obtain

$$\frac{\partial\rho(\delta\mathbf{\Gamma},t)}{\partial t}=-\sum_{\alpha=1}^{N}\frac{\delta p_\alpha}{m}\frac{\partial\rho(\delta\mathbf{\Gamma},t)}{\partial\delta q_\alpha}+\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\sum_{\mu=1}^{N}\times\sum_{\nu=1}^{N}\frac{1}{2}D_{\alpha\mu\beta\nu}\delta q_\alpha\delta q_\beta\frac{\partial^2\rho(\delta\mathbf{\Gamma},t)}{\partial\delta p_\mu\,\partial\delta p_\nu}. \tag{13}$$

Equation (13) is a master equation for the tangent vector $\delta\mathbf{\Gamma}=(\delta\mathbf{q},\delta\mathbf{p})^T$ with $\delta\mathbf{q}\equiv(\delta q_1,\delta q_2,\ldots,\delta q_N)^T$ and $\delta\mathbf{p}\equiv(\delta p_1,\delta p_2,\ldots,\delta p_N)^T$, especially the type of the equation called the Fokker-Plank equation because its right-hand side includes up to the second derivative of the probability density with respect to the variables. The derivation of Eq. (13) is given in Appendix A. We should notice that the first term on the right-hand side of Eq. (13) has the same form as the corresponding term in the Liouville equation describing noninteracting particle systems, because the dynamics of the phase space point and the tangent space point coincide in the noninteracting particle system. The characteristics of the system are introduced through the fourth rank tensor $D_{\alpha\mu\beta\nu}$ and a boundary condition for a solution of Eq. (13).

III. LYAPUNOV SPECTRUM IN THE RANDOM MATRIX APPROACH

In this section, using the Fokker-Plank equation (13) for the tangent vector we investigate the absolute values of the tangent vectors, whose asymptotic behaviors lead to the Lyapunov exponents.

We introduce the quantities $\Upsilon_{jk}^{(l)}(t)$, $l=1,2,3$ as

$$\Upsilon_{jk}^{(1)}(t)\equiv\langle\delta q_j\delta q_k\rangle_t, \tag{14}$$

$$\Upsilon_{jk}^{(2)}(t)\equiv\langle(\delta q_j\delta p_k+\delta q_k\delta p_j)\rangle_t/2, \tag{15}$$

$$\Upsilon_{jk}^{(3)}(t)\equiv\langle\delta p_j\delta p_k\rangle_t, \tag{16}$$

where $\langle\cdots\rangle_t$ denotes the average over the probability density $\rho(\delta\mathbf{\Gamma},t)$ at time t: $\langle\cdots\rangle_t\equiv\int d\delta\mathbf{\Gamma}\rho(\delta\mathbf{\Gamma},t)\cdots$. These quantities consist of elements of the averaged matrix $\langle\delta\mathbf{\Gamma}\delta\mathbf{\Gamma}^T\rangle_t$, and satisfies the condition

$$\Upsilon_{jk}^{(l)}(t)=\Upsilon_{kj}^{(l)}(t). \tag{17}$$

Equation (13) connects these quantities as

$$\frac{d\Upsilon_{jk}^{(1)}(t)}{dt}=\frac{2}{m}\Upsilon_{jk}^{(2)}(t), \tag{18}$$

$$\frac{d\Upsilon_{jk}^{(2)}(t)}{dt}=\frac{1}{m}\Upsilon_{jk}^{(3)}(t), \tag{19}$$

$$\frac{d\Upsilon_{jk}^{(3)}(t)}{dt}=\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}D_{j\alpha k\beta}\Upsilon_{\alpha\beta}^{(1)}(t), \tag{20}$$

where we assumed the probability density $\rho(\delta\mathbf{\Gamma},t)$ to be zero at the boundary of the tangent space, and used Eqs. (11), (12), and (17) to derive Eq. (20).

Now we introduce a fourth rank tensor T_{jkln} satisfying the condition

$$\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}T_{\alpha jk\beta}T_{\beta nl\alpha}=\delta_{jl}\delta_{kn}, \tag{21}$$

so that we obtain

$$\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\sum_{\mu=1}^{N}\sum_{\nu=1}^{N}T_{j\alpha\beta k}D_{\alpha\mu\nu\beta}T_{n\nu\mu l}=\Lambda_{jk}\delta_{jl}\delta_{kn}, \tag{22}$$

with the real second rank tensor Λ_{jk} satisfying the condition $\Lambda_{jj}\geqslant 0$ [38]. [Concerning Eqs. (21) and (22), for example, the existence of the real tensor Λ_{jk} satisfying Eq. (22) simply comes from the fact that by the condition (12) we can regard the quantity D_{jkln} as the matrix element $\mathcal{D}_{\gamma_1(j,n)\gamma_2(k,l)}$ of the $N^2\times N^2$ real symmetric matrix $\mathcal{D}\equiv(\mathcal{D}_{\gamma_1(j,n)\gamma_2(k,l)})$ where $\gamma_n=\gamma_n(j,k)$, $n=1,2$ are functions from $j\in\{1,2,\ldots,N\}$ and $k\in\{1,2,\ldots,N\}$ to $\gamma_n\in\{1,2,\ldots,N^2\}$.] We will discuss an example of the tensors T_{jkln} and Λ_{jk} in the following section. By using the tensor T_{jkln} we transform the quantities $\Upsilon_{jk}^{(l)}(t)$ to $\tilde{\Upsilon}_{jk}^{(l)}(t)$ as

$$\tilde{\Upsilon}_{jk}^{(l)}(t)\equiv\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}T_{j\alpha\beta k}\Upsilon_{\alpha\beta}^{(l)}(t). \tag{23}$$

The inverse transformation to derive the quantity $\Upsilon_{jk}^{(l)}(t)$ from the quantity $\tilde{\Upsilon}_{\alpha\beta}^{(l)}(t)$ is simply given by

$$\Upsilon_{jk}^{(l)}(t)=\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}T_{\beta kj\alpha}\tilde{\Upsilon}_{\alpha\beta}^{(l)}(t) \tag{24}$$

noting the relation (21). Especially we should notice the relations $\langle|\delta\mathbf{q}|^2\rangle_t=\Sigma_{\alpha=1}^N\tilde{Y}_{\alpha\alpha}^{(1)}(t)$ and $\langle|\delta\mathbf{p}|^2\rangle_t=\Sigma_{\alpha=1}^N\tilde{Y}_{\alpha\alpha}^{(3)}(t)$ under the condition $\Sigma_{\alpha=1}^N T_{j\alpha\alpha k}=\delta_{jk}$.

Noting that the quantity $\tilde{Y}_{jj}^{(1)}(t)$ is the square of the amplitude of the jth spatial component of an orthogonal-transformed tangent vector, we introduce the positive (or zero) Lyapunov exponents λ_j as

$$\lambda_j=\lim_{t\to\infty}\frac{1}{2t}\ln\frac{\tilde{Y}_{jj}^{(1)}(t)}{\tilde{Y}_{jj}^{(1)}(0)}, \tag{25}$$

namely, as the exponential rate of the time evolution of the average of magnitude of the infinitesimal deviation of the phase space orbit in the long time limit. The introduction of Lyapunov exponents by the spatial coordinate part only (or the momentum part only) of the tangent vector has been used previously in Refs. [26,39].

As shown in Appendix B, the Lyapunov exponents are simply given by

$$\lambda_j=\left[\frac{\Lambda_{jj}}{(2m)^2}\right]^{1/3}, \tag{26}$$

using the quantity Λ_{jj} introduced in Eq. (22). Equation (26) is the key result of this paper. This equation connects the Lyapunov exponents directly with the tensor D_{jkln} representing the strength of correlation of the particle interactions, and also shows the fact that in a system described by the random matrix dynamics the Lyapunov exponents are independent of the initial condition like in the deterministic chaos. It should be noted that the Lyapunov exponents λ_j given by Eq. (26) are the same with the quantities derived from the equations $\lim_{t\to\infty}(2t)^{-1}\ln[\tilde{Y}_{jj}^{(k)}(t)/\tilde{Y}_{jj}^{(k)}(0)]$, $k=2,3$ in this approach (see Appendix B).

We sort the Lyapunov exponents $\lambda_1,\lambda_2,\ldots,\lambda_N$ so that they contain a decreasing (or equal) sequence, and introduce the set $\{\lambda^{[1]},\lambda^{[2]},\ldots,\lambda^{[N]}\}$ satisfying the condition

$$\{\lambda^{[1]},\lambda^{[2]},\ldots,\lambda^{[N]}\}=\{\lambda_1,\lambda_2,\ldots,\lambda_N\}, \tag{27}$$

$$\lambda^{[1]}\geqslant\lambda^{[2]}\geqslant\cdots\geqslant\lambda^{[N]}. \tag{28}$$

This sorted set of the Lyapunov exponents is called the Lyapunov spectrum.

IV. A SIMPLIFICATION

Before considering the Lyapunov spectra in concrete models using the formula given in the preceding section, we discuss an assumption to simplify the model calculations. We consider the case where all matrix elements $R_{jk}(t)$, $j=1,2,\ldots,N$ and $k=1,2,\ldots,N$ have the same time dependence, namely,

$$R_{jk}(t)=r(t)A_{jk}, \tag{29}$$

where $r(t)$ is a normalized Gaussian white randomness in the sense of $\langle r(t_1)r(t_2)\cdots r(t_{2n-1})\rangle=0$ and $\langle r(t_1)r(t_2)\cdots r(t_{2n})\rangle=\Sigma_{P_d}\delta(t_{j_1}-t_{j_2})\delta(t_{j_3}-t_{j_4})\cdots\delta(t_{j_{2n-1}}-t_{j_{2n}})$ for any integer n and $A\equiv(A_{jk})$ is a time-independent $N\times N$ matrix. The matrix A must be a symmetric matrix,

$$A_{kj}=A_{jk}, \tag{30}$$

because the matrix $R(t)$ is symmetric.

The assumption (29) simplifies our considerations, because in this case the tensor D_{jkln} is represented as the multiplication of the matrix element A_{jk} with A_{ln}. Thus

$$D_{jkln}=A_{jk}A_{ln} \tag{31}$$

and the conditions (11) and (12) for the tensor D_{jkln} are automatically satisfied under the condition (30). The condition (30) also implies that the matrix A is diagonalizable using an orthogonal matrix $U\equiv(U_{jk})$ satisfying $U^TU=UU^T=I$, namely, $(U^TAU)_{jk}=a_j\delta_{jk}$ with real eigenvalues a_l, $l=1,2,\ldots,N$ of the matrix A. The fourth rank tensor T_{jkln} is constructed as

$$T_{jkln}=U_{kj}U_{ln}, \tag{32}$$

which satisfies the conditions (21), (22), and $\Sigma_{\alpha=1}^N T_{j\alpha\alpha k}=\delta_{jk}$. Here the second rank tensor Λ_{jk} is represented as

$$\Lambda_{jk}=a_ja_k, \tag{33}$$

which satisfies the conditions $\Lambda_{jj}\geqslant0$. Equations (26) and (33) lead to the expression of the Lyapunov exponents as

$$\lambda_j=\left|\frac{a_j}{2m}\right|^{2/3}. \tag{34}$$

After all, under the assumption (29) the calculation of the Lyapunov spectrum is attributed to the eigenvalue problem of the matrix A.

V. ONE-DIMENSIONAL MODELS AND STEPS OF THEIR LYAPUNOV SPECTRA

In this section we consider simple one-dimensional models and calculate their Lyapunov spectra using the formula given in Sec. III under the assumptions discussed in Sec. IV. We are generally interested in which ingredients of the model system lead to particular features of the Lyapunov spectra. We are especially interested in models that satisfy the total momentum conservation and show stepwise structures in their Lyapunov spectra. Numerically the observation of Lyapunov modes is associated with the systems with a stepwise structure of their Lyapunov spectrum, such as the many-hard-disk system of Refs. [14,15,19,32,33].

We construct a model consisting of N particles in a one-dimensional space. In this case the off-diagonal matrix element A_{jk}, $j\neq k$ represents the strength of the interaction between the jth particle and the kth particle. The diagonal matrix element A_{jj} is determined by the total momentum conservation that imposes the condition

$$\sum_{k=1}^N A_{jk}=0 \tag{35}$$

for the matrix A. Equation (35) is derived from Eq. (29) and the equation $\Sigma_{k=1}^{N} R_{jk}(t)=0$ satisfied at any time t. Using the formula (34) this condition implies that the N dimensional vector, whose components are equal, is an eigenstate of the matrix A corresponding to the eigenvalue 0, so we obtain

$$\lambda^{[N]}=0, \tag{36}$$

namely, there is a zero Lyapunov exponent corresponding to the conservation of total momentum.

A. One-dimensional model with a stepwise structure of its Lyapunov spectrum

As a first step, we consider the case where each particle interacts only with its nearest-neighbor particles with the same strength of interaction. We impose periodic boundary conditions, namely, that the particles are on a one-dimensional ring structure. This situation is described by the matrix $A=A^{(0)}\equiv(A_{jk}^{(0)})$ defined by

$$A_{jk}^{(0)}\equiv\omega[-2\delta_{jk}+\delta_{j(k+1)}+\delta_{(j+1)k}+\delta_{j(k-N+1)}+\delta_{k(j-N+1)}] \tag{37}$$

with a (nonzero) real constant ω. We can calculate the eigenvalues of the matrix $A^{(0)}$ (see Appendix C for the calculation details), and obtain the Lyapunov exponents $\lambda_n=\lambda_n^{(0)}$ as

$$\lambda_n^{(0)}=\left[\frac{|\omega|}{m}\left(1-\cos\frac{2\pi n}{N}\right)\right]^{2/3}. \tag{38}$$

It is important to note that the Lyapunov exponents given by Eq. (38) satisfy the conditions $\lambda_N^{(0)}=0$ and

$$\lambda_j^{(0)}=\lambda_{N-j}^{(0)} \tag{39}$$

for $j<N/2$, namely, the Lyapunov spectrum has degeneracies. In other words the spectrum has a stepwise structure. The maximum Lyapunov exponent is given by $(2|\omega|/m)^{2/3}$ when N is even or $\{|\omega|[1+\cos(\pi/N)]/m\}^{2/3}$ when N is odd.

B. Robustness of the stepwise structure of the Lyapunov spectrum to a perturbation

Next we consider the case where each particle interacts with its nearest-neighbor particles by slightly different interaction strengths. This situation is described by the matrix $A=A^{(0)}+\varepsilon A^{(1)}$ with ε a small parameter and the matrix $A^{(1)}$ defined as

$$A_{jk}^{(1)}=-(\chi_j+\chi_{j+1})\delta_{jk}+\chi_j\delta_{j(k+1)}+\chi_k\delta_{k(j+1)} +\chi_1[\delta_{j(k-N+1)}+\delta_{k(j-N+1)}], \tag{40}$$

where χ_j, $j=1,2,\ldots,N+1$ are real constants satisfying the condition $\chi_{N+1}=\chi_1$. In this case we can calculate the eigenvalues of this matrix A to first order in the small parameter ε, and obtain the Lyapunov exponents $\lambda_n=\lambda_n^{(0)}+\varepsilon\lambda_n^{(1)}+O(\varepsilon^2)$ in the expanded form by the parameter ε with the quantities

$$\lambda_j^{(1)}=\frac{2\lambda_j^{(0)}}{3\omega}(\tilde{\chi}_0+|\tilde{\chi}_j|), \tag{41}$$

$$\lambda_{N-j}^{(1)}=\frac{2\lambda_{N-j}^{(0)}}{3\omega}(\tilde{\chi}_0-|\tilde{\chi}_j|), \tag{42}$$

in the case of $j<N/2$, where $\tilde{\chi}_j$ is defined by

$$\tilde{\chi}_j\equiv\frac{1}{N}\sum_{\alpha=1}^{N}\chi_\alpha\exp\left(\frac{4\pi i\alpha j}{N}\right). \tag{43}$$

[See Appendix C for the derivations of Eqs. (41) and (42).] Here we numbered so that we obtain $\omega(\lambda_j^{(1)}-\lambda_{N-j}^{(1)})\geqslant 0$. It may be noted that the quantity $\tilde{\chi}_0$ in Eqs. (41) and (42) is simply the arithmetic average of the coefficients χ_j, $j=1,2,\ldots,N$: $\tilde{\chi}_0=N^{-1}\Sigma_{j=1}^{N}\chi_j$. We also obtain $\lambda_{N/2}^{(1)}=2\lambda_{N/2}^{(0)}\tilde{\chi}_0/(3\omega)$ if the number N is even, and $\lambda_N^{(1)}=0$ for the first order correction to the nondegenerate Lyapunov exponents.

Equations (41) and (42) tell us about the robustness of the stepwise structure of the Lyapunov spectrum appearing in the nonperturbed system against the perturbation. To discuss this point we consider the deviation $|\lambda_j-\lambda_{N-j}|$ of the two points in the jth step of the Lyapunov spectrum, which is given by

$$|\lambda_j-\lambda_{N-j}|=\frac{4}{3}\left|\frac{\varepsilon\tilde{\chi}_j}{\omega}\right|\lambda_j+O(\varepsilon^2) \tag{44}$$

in the case of $j<N/2$. Equation (44) implies that the degeneracies, namely, the stepwise structure of the Lyapunov spectrum is removed by the perturbation $\varepsilon A^{(1)}$, but their deviations are small in a region of small Lyapunov exponents rather than in a region of large Lyapunov exponents, as far as the quantity $\tilde{\chi}_j$ is almost j independent. In other words, this consideration suggests that the stepwise structure of the Lyapunov spectrum is robust in the region of small Lyapunov exponents. It should be emphasized that this is consistent with the numerical results of the Lyapunov spectrum in a many-hard-disk model, which shows the stepwise structure of the Lyapunov spectrum only in a region of small Lyapunov exponents [14,15,19,32,33]. By using Eqs. (41) and (42) we can also discuss a perturbational effect in a global shape of the Lyapunov spectrum. Equations (41) and (42) lead to the shift of the jth step of the Lyapunov spectrum as

$$\frac{\lambda_j+\lambda_{N-j}}{2}-\lambda_j^{(0)}=\frac{2\tilde{\chi}_0}{3\omega}\lambda_j\varepsilon+O(\varepsilon^2) \tag{45}$$

in the case $j<N/2$. Equation (45) implies that in the case of $\tilde{\chi}_0\varepsilon/\omega>0$ ($\tilde{\chi}_0\varepsilon/\omega<0$) the perturbation $\varepsilon A^{(1)}$ makes the slope of the Lyapunov spectrum more (less) steep than in the nonperturbed case.

Now we discuss the robustness of the stepwise structure of the Lyapunov spectrum against perturbation in a slightly different way, in which the fluctuations of the matrix ele-

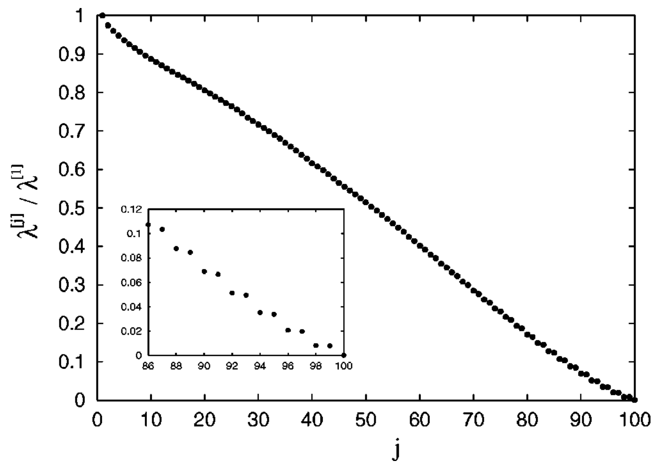

FIG. 1. Lyapunov spectra of a one-dimensional system with nearest-neighbor interactions in the case of randomly chosen matrix elements. The inset shows an enlarged graph of a small Lyapunov exponent region.

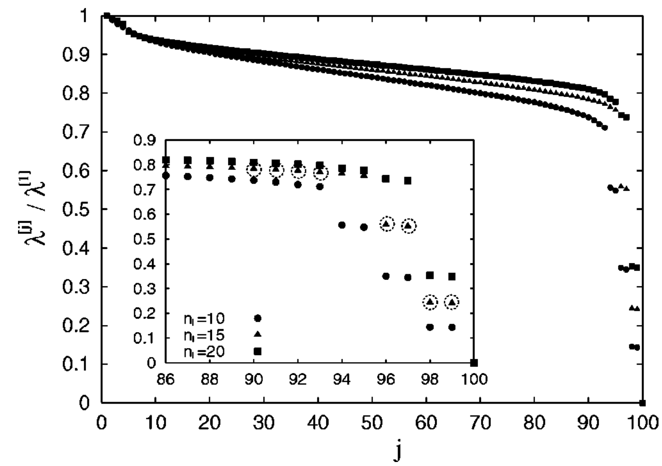

FIG. 2. Lyapunov spectra of one-dimensional systems with different length n_l of long range interactions in the case of $N=100$. The three graphs correspond to the cases $n_l=10$ (the circular dots), $n_l=15$ (the triangular dots), and $n_l=20$ (the square dots), respectively. The inset shows enlarged graphs of a small Lyapunov exponent region.

ments of A are large enough so that the above first order perturbed discussion is no longer correct. We consider a one-dimensional system described by the matrix $A=A^{(1)}$ given by Eq. (40), in which the quantities χ_j, $j=1,2,\dots,N$ are chosen randomly from the region (a,b). Figure 1 is a Lyapunov spectrum normalized by the maximum Lyapunov exponent in such a system consisting of 100 particles ($N=100$). Here, we chose the region (a,b) as $a=\omega/2$ and $b=3\omega/2$, and took the arithmetic average over this randomness for the normalized Lyapunov spectrum. Although the magnitude $b-a(=\omega)$ of fluctuations in the matrix elements of the matrix A is of the same scale as the averaged magnitude $(a+b)/2(=\omega)$ of the matrix elements of the matrix A, we can still recognize some steps in the Lyapunov spectrum in the region of small Lyapunov exponents. It is interesting to note that in such a case the global shape of the Lyapunov spectrum is rather close to a straight line, like discussed in Refs. [4–8].

C. Effect of long range interactions and wavelike structures in eigenstates

In this subsection we consider the effects of long range interactions between particles on the Lyapunov spectrum. The effect of the long range interactions between particles can be taken into account using the matrix $A=A^{[n_l]}\equiv(A_{jk}^{[n_l]})$ defined by

$$A_{jk}^{[n_l]}=\sum_{l=1}^{n_l}\,[-(\sigma_j^{[l]}+\sigma_{j+l}^{[l]})\delta_{jk}+\sigma_j^{[l]}\delta_{j(k+l)}+\sigma_k^{[l]}\delta_{k(j+l)} +\sigma_{j-N+l}^{[l]}\delta_{j(k+N-l)}+\sigma_{k-N+l}^{[l]}\delta_{k(j+N-l)}], \qquad (46)$$

with real constants $\sigma_j^{[l]}$, where $n_l(<N/2)$ is the length of the interactions and its value means that the particles can interact with up to their n_lth nearest-neighbor particles. As shown in Appendix D, we can obtain an analytical expression for the Lyapunov spectrum derived from the matrix (46) if all of the nonzero matrix elements $\sigma_j^{[l]}$, $j=1,2,\dots,N$, $l=1,2,\dots,n_l$ take the same value.

In this subsection we consider the case where the matrix element $\sigma_j^{[l]}$ is chosen randomly, like in the model discussed at the end of the preceding subsection. Figure 2 is the Lyapunov spectrum normalized by the maximum Lyapunov exponent in the system described by the matrix $A=A^{[n_l]}$ with the quantities $\sigma_j^{[l]}$ chosen randomly from the region $(\omega/2,3\omega/2)$ for the case of $n_l=10$, 15, and 20 in the system consisting of 100 particles ($N=100$). (This randomness in the quantities $\sigma_j^{[l]}$ is adopted to draw all of the figures hereafter in this subsection.) In these graphs we took the arithmetic average of the Lyapunov spectra over the randomness of the quantities $\sigma_j^{[l]}$. This figure shows that the long range interactions separate the Lyapunov spectrum clearly into a part exhibiting stepwise structure and a part changing smoothly. (The randomness of the quantities $\sigma_j^{[l]}$ is not essential for this separation in the Lyapunov spectrum. It mainly plays the role of smoothening the Lyapunov spectrum in the region of large Lyapunov exponents.) The stepwise structure of the Lyapunov spectrum appears only in a region of small Lyapunov exponents, as suggested in the preceding subsection.

The stepwise region of the Lyapunov spectrum becomes smaller and smaller as the interaction length n_l of particles increases. (As a limit of the long range interactions, as shown in Appendix D, we can easily show that in the system, where each particle interacts with all the other particles with the same strength, all of the positive Lyapunov exponents take the same value.) Besides, the heights of the steps of the Lyapunov spectrum are approximately proportional to the length n_l of the interactions. These characteristics are clearly evident in Fig. 3, showing the length n_l dependence of the jth positive Lyapunov exponents $\lambda^{[j]}$, $j=91,93,95,97,99$ in the case of 100 particles. In these graphs we also took the arithmetic average of the Lyapunov exponents over the ran-

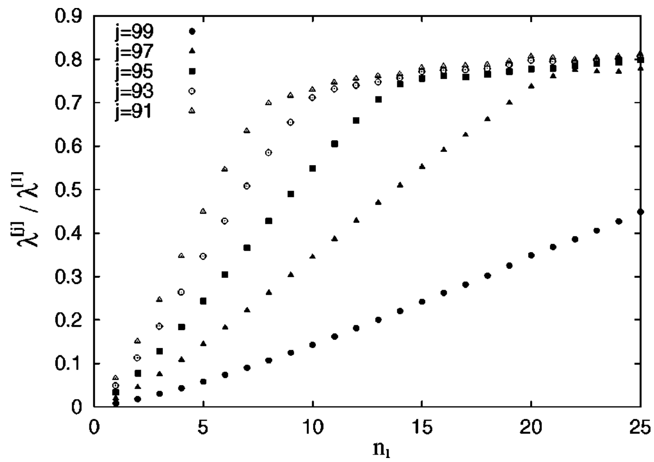

FIG. 3. Dependence of the long range interaction length n_l on the Lyapunov exponents $\lambda^{[j]}/\lambda^{[1]}$ in the case of $N=100$. The graphs with the circular dots, the triangular dots, the square dots, the open circular dots, and the open triangular dots correspond to the case of $j=99,97,95,93$, and 91, respectively.

domness of the quantities $\sigma_j^{[l]}$. This figure shows that the values of small Lyapunov exponents increase with increase in the interaction length n_l as long as they are in the region of the steps of the Lyapunov spectrum, and if the interaction length n_l is bigger than a critical value then their n_l dependences are slowed down, meaning that they are in the region of the Lyapunov spectrum that is changing smoothly.

It may be interesting to investigate the difference between the two parts of the Lyapunov spectrum from the point of view of the eigenstate. Figure 4 is the real eigenstate of the matrix $A^{[n_l]}$ in the case of $n_l=15$ and $N=100$ with randomly chosen quantities $\sigma_j^{[l]}$. (Note that we did not take the average over the randomness of the quantities $\sigma_j^{[l]}$ to draw this figure.) The graphs (a), (b), (c), and (d) are the eigenstates corresponding to the Lyapunov exponents $\lambda^{[j]}$ for $j=99,98$, $j=97,96$, $j=93,92$, and $j=91,90$, respectively. These Lyapunov exponents correspond to the triangular dots surrounded by the broken lines in the inset of Fig. 2. We can see clear wavelike structures of approximately sinusoidal type in the graphs (a) and (b), which correspond to the Lyapunov exponents composing the steps in the Lyapunov spectrum. The wavelength of the waves in the graph (b) is half of the wavelength of the waves in the graph (a). On the other hand, we cannot recognize such a wavelike structure in the graphs (c) and (d), which belongs to the Lyapunov exponents in the part of the Lyapunov spectrum which is changing smoothly.

It should be noted that the wavelike structure of the eigenstates already appears even in the system described by the matrix $A=A^{(0)}$ defined by Eq. (37) with nearest-neighbor interactions of a constant strength. Therefore the point is that such a wavelike structure of the eigenstates is not destroyed by the long range interactions and the random interaction strengths only in the eigenstates that correspond to the small positive Lyapunov exponents in the steps of the Lyapunov spectrum.

One may regard the long range interactions discussed in this subsection as a kind of high dimensional effect, because if particles move in a two- or three-dimensional space then they can interact with more than two particles, even in hard-core interactions. However, some high dimensional effects, for example, the total momentum conservation in each orthogonal direction and so on, are still missing in the discus-

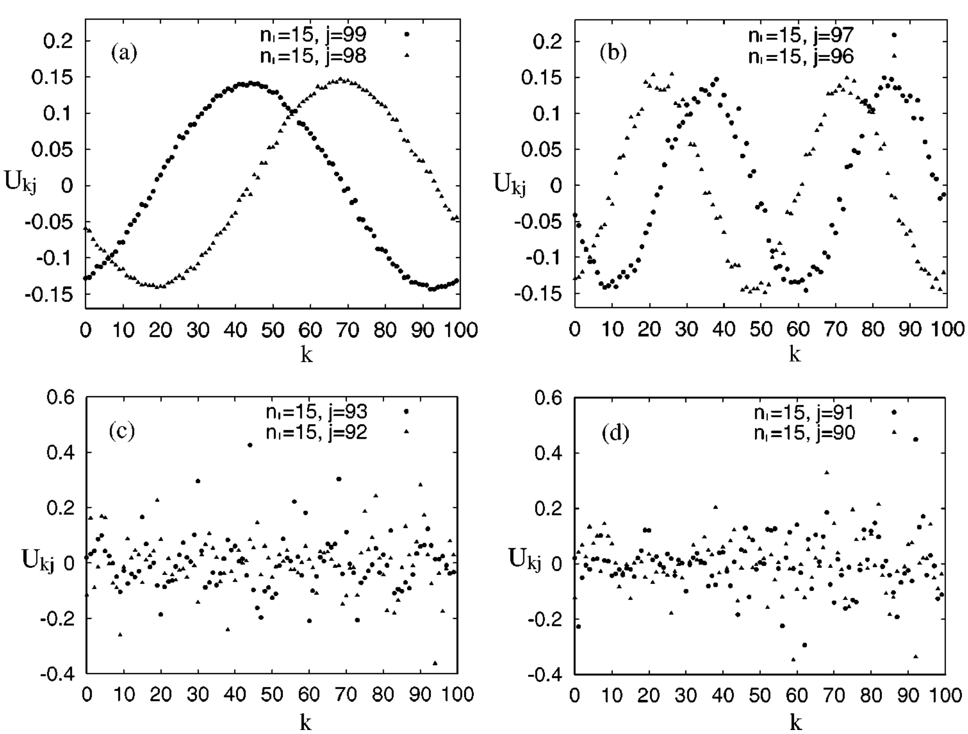

FIG. 4. Eigenstates of matrix $A^{[n_l]}$ in the case of $n_l=15$ and $N=100$: the graph (a) ($j=98,99$), the graph (b) ($j=96,97$), the graph (c) ($j=92,93$), and the graph (d) ($j=90,91$) for the Lyapunov exponent $\lambda^{[j]}$. These Lyapunov exponents correspond to the triangular dots surrounded by the broken lines in the inset of Fig. 2.

sions in this subsection. In the following section we consider high dimensional effects more explicitly.

VI. TWO-DIMENSIONAL MODEL WITH A TRIANGULAR LATTICE STRUCTURE

In the one-dimensional models discussed in the preceding section, the steps of the Lyapunov spectrum are caused by the periodic boundary conditions, and all of them consist of only two points. In this section we consider a two-dimensional model, and show that it is possible to get wider steps than in the one-dimensional model as a high dimensional effect. Specifically, we concentrate on a model that has steps consisting of four and eight points, which are actually found in a square system consisting of many hard disks [32].

In the two-dimensional system, the position of each particle is specified by a two-dimensional vector $\mathbf{q}_j$, $j=1,2,\ldots,\tilde{N}$, where $\tilde{N}$ is the number of the particles and $N=2\tilde{N}$. In such a case the matrix A is a $(2\tilde{N})\times(2\tilde{N})$ matrix, and can be represented as a block matrix consisting of 2×2 matrices $B^{(jk)}$, $j=1,2,\ldots,\tilde{N}$ and $k=1,2,\ldots,\tilde{N}$. Here $B^{(jk)}\equiv(B^{(jk)}_{j'k'})$ is given by

$$B^{(jk)}\equiv\begin{pmatrix} A_{(2j-1)(2k-1)} & A_{(2j-1)(2k)} \\ A_{(2j)(2k-1)} & A_{(2j)(2k)} \end{pmatrix} \tag{47}$$

corresponding to the matrix $-\partial^2 V/\partial\mathbf{q}_j\,\partial\mathbf{q}_k$, and its components represent the strengths of the interactions between components of positions of the jth particle and the kth particle. The matrix $B^{(jk)}$ is symmetric and the matrix A is also symmetric in the sense of the block matrix, namely,

$$B^{(jk)}_{j'k'}=B^{(jk)}_{k'j'}=B^{(kj)}_{j'k'}. \tag{48}$$

The diagonal block $B^{(jj)}$ is determined by the condition of the total momentum conservation

$$\sum_{k=1}^{\tilde{N}} B^{(jk)}=0, \tag{49}$$

which comes from Eq. (29) and the relation $\Sigma_{k=1}^{\tilde{N}}\partial^2 V/\partial\mathbf{q}_j\,\partial\mathbf{q}_k=0$ and is simply the two-dimensional version of the condition (35). This condition (49) implies that the Lyapunov exponents include at least two zero components:

$$\lambda^{[N-1]}=\lambda^{[N]}=0 \tag{50}$$

corresponding to the conservation of the total momentum.

To construct a two-dimensional model it is convenient to use a lattice picture, because in the approach of this paper, under the assumption discussed in Sec. IV, the model is specified by pairs of interacting particles and their interaction strengths. In such a lattice, each lattice site corresponds to a particle and a connection between sites means that particles on those two sites can interact with each other as nearest-neighbor particles. Now we consider an equilateral triangular lattice from such a point of view. This situation is motivated by the fact that in Ref. [32] the system consists of many disks that pack in a two-dimensional space as a hexagonal close-packed structure in the high density limit. The triangular lattice has three directions to connect the sites, and arrange the triangular lattice so that it fits within a square where one of the sides of the square is parallel to one of the three directions of the triangular lattice, as shown in Fig. 5(a) where such parallel lines are horizontal. For simplicity we assume that the number of lattice sites in each horizontal line is equal and is given by $N_1(>1)$. We put the number of such lines as N_2, which should be roughly given by $N_2\approx(2/\sqrt{3})N_1$.

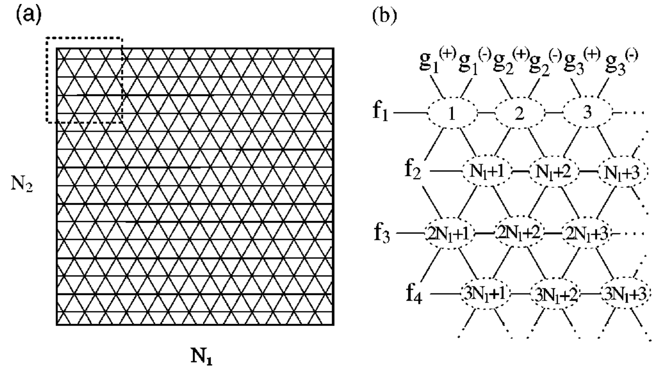

FIG. 5. (a) The triangular lattice system with a square shape. N_1 is the number of particles in a horizontal line, and N_2 is the number of horizontal lines. (b) Numbering of the particles and the boundary condition in the part surrounded by the square of broken line in (a).

Next, we introduce the boundary condition for this triangular lattice model. For this purpose we assign numbers $1,2,\ldots,N_1$ to the lattice sites (namely, particles) in the first horizontal row of particles from left to right, and numbers $N_1+1,N_1+2,\ldots,2N_1$ in the second row and so on until the last row numbered $(N_2-1)N_1+1,(N_2-1)N_1+2,\ldots,N_2N_1$, as shown in Fig. 5(b). We define the number f_{2j-1} as the number of the particle that has nearest-neighbor interactions with the $[2(j-1)N_1+1]$th particle, and the number f_{2j} as the number of the particle that has nearest-neighbor interactions with the $[2(j-1)N_1+1]$th, $[(2j-1)N_1+1]$th, and $[2jN_1+1]$th particles ($j=1,2,\ldots,\mathrm{Int}\{N_2/2\}$). Here, $\mathrm{Int}\{x\}$ means to take the integer part of x for any real number x. We also define the numbers $g_j^{(+)}$ and $g_j^{(-)}$ as the numbers of the particles that have nearest-neighbor interactions with the jth particle through the upper-left line and the upper-right line, respectively ($j=1,2,\ldots,N_1$). (See Fig. 5(b) for the definitions of the numbers f_j and $g_j^{(\pm)}$.) The number f_j specifies the interaction between the left side and the right side of the square, and the numbers $g_j^{(\pm)}$ specifies the interaction between the upper side line and the lower side of the square. In this section we consider the case that these numbers are given by

$$f_j=jN_1, \tag{51}$$

$$g_j^{(\pm)}=(N_1-1)N_2+\min_{k\in\{1,2,\ldots\}}\{h_{jk}^{(\pm)};h_{jk}^{(\pm)}>0\}, \tag{52}$$

where $h_{jk}^{(\pm)}$ is defined by

$$h_{jk}^{(\pm)} = j \pm \mathrm{Int}\left\{\frac{1}{2}\left(N_2 - \frac{1\pm 1}{2}\right)\right\} \mp kN_1 . \tag{53}$$

The boundary conditions (51) and (52) imply that every line of the three directions in the triangular lattice is periodic. We assume the condition $g_j^{(+)} \neq g_j^{(-)}$ so that every particle has the six nearest-neighbor particles.

Before giving the matrix A of the triangular lattice model including a long range interaction, we consider the matrix $A=\widetilde{A}_0$ including only the effect of the nearest-neighbor interactions of a constant strength. We define the $2N_1\times 2N_1$ matrices C_l, $l=1,2,3,4$, each of which is a block matrix consisting of 2×2 matrices $C_l^{(jk)}$, $j=1,2,\ldots,N_1$, $k=1,2,\ldots,N_1$, by

$$C_l^{(jk)} \equiv -6\mathcal{B}\delta_{jk} + \mathcal{B}\,[\,\delta_{j(k+1)} + \delta_{k(j+1)} + \delta_{j(k-N_1+1)} + \delta_{k(j-N_1+1)}], \tag{54}$$

$$C_2^{(jk)} \equiv \mathcal{B}\,[\,\delta_{jk} + \delta_{j(k+1)} + \delta_{j(k-N_1+1)}], \tag{55}$$

$$C_3^{(jk)} \equiv \mathcal{B}\,[\,\delta_{jk} + \delta_{k(j+1)} + \delta_{k(j-N_1+1)}], \tag{56}$$

$$C_4^{(jk)} \equiv \mathcal{B}\{\delta_{k[g_j^{(+)}-(N_1-1)N_2]} + \delta_{k[g_j^{(-)}-(N_1-1)N_2]}\}, \tag{57}$$

where $\mathcal{B}$ is a 2×2 matrix. The matrix $\widetilde{A}_0$ is introduced as the block matrix defined by

$$\widetilde{A}_0 = \begin{pmatrix} C_1 & C_2 & \overline{0} & \overline{0} & \overline{0} & \cdots & \overline{0} & C_4 \\ C_2^T & C_1 & C_3 & \overline{0} & \overline{0} & \cdots & \overline{0} & \overline{0} \\ \overline{0} & C_3^T & C_1 & C_2 & \overline{0} & & & \\ \overline{0} & \overline{0} & C_2^T & C_1 & C_3 & & & \\ \overline{0} & \overline{0} & \overline{0} & C_3^T & C_1 & \ddots & & \\ \vdots & \vdots & & & \ddots & \ddots & & \\ \overline{0} & \overline{0} & & & & & C_1 & C_{\tilde{\nu}} \\ C_4^T & \overline{0} & & & & & C_{\tilde{\nu}}^T & C_1 \end{pmatrix}, \tag{58}$$

with the $2N_1\times 2N_1$ null matrix $\overline{0}$, where $\tilde{\nu}$ is 2 (3) if N_2 is an odd (even) number.

The matrix $A=\widetilde{A}^{[n_l]}$ of the triangular lattice model including the effect of the long range interactions up to the n_lth nearest neighbor interactions is simply given as follows. We consider the matrix $\widetilde{\mathcal{A}}_0'$ given by the matrix $\widetilde{A}_0$ except that the matrix $\mathcal{B}$ in the matrix $\widetilde{A}_0$ are replaced by the 2×2 matrices whose matrix elements are positive constants. By using such a $(2N_1N_2)\times(2N_1N_2)$ matrix $\widetilde{\mathcal{A}}_0'$ we calculate the n_l times multiplication $(\widetilde{\mathcal{A}}_0')^{n_l}$ of the matrix $\widetilde{\mathcal{A}}_0'$. The nonzero elements of the matrix $\widetilde{A}^{[n_l]}$ are equivalent to the nonzero elements of the matrix $(\widetilde{\mathcal{A}}_0')^{n_l}$. After determining the nonnull upper off-diagonal blocks $B^{(jk)}$, $j<k$ of the matrix $\widetilde{A}^{[n_l]}$ satisfying the condition (48) by such a process, the diagonal block $B^{(jj)}$ and the lower off-diagonal blocks $B^{(jk)}$, $j>k$ of the matrix $\widetilde{A}^{[n_l]}$ are determined so that the matrix $\widetilde{A}^{[n_l]}$ is symmetric and satisfies the condition (49). It should be noted that we could use a similar method to obtain the matrix $A^{[n_l]}$ including the effect of long range interactions in the one-dimensional model in the preceding section.

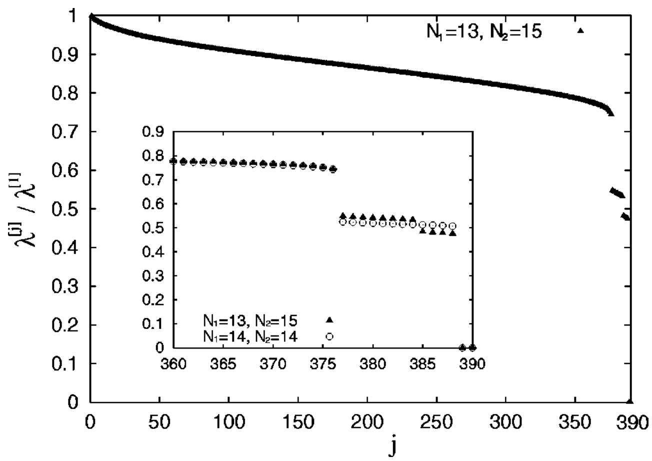

FIG. 6. Lyapunov spectrum of the triangular lattice system in the case of $N_1=13$, $N_2=15\approx(2/\sqrt{3})N_1$, and $n_l=3$ (the triangular dots). The inset shows enlarged graphs of the part including the stepwise structures in the Lyapunov spectra in the case of $N_1=13$ and $N_2=15$ (the triangular dots) and the case of $N_1=N_2=14$ (the circular dots) for a comparison. The Lyapunov spectrum in the case of $N_1=N_2=14$ is shifted by -2 in the j direction, so that the positions of the zero Lyapunov exponents coincide in both the cases.

We restrict our consideration in the case that the 2×2 matrix $B^{(jk)}$, which is the block element of the matrix A in the two-dimensional system, is diagonalized,

$$B_{12}^{(jk)} = B_{21}^{(jk)} = 0. \tag{59}$$

This means that two components of the position of each particle do not interact with each other.

Now we calculate the Lyapunov spectrum for such a matrix $\widetilde{A}^{[n_l]}$ by using the formula (34). Figure 6 is the Lyapunov spectrum normalized by the maximum Lyapunov exponent in such a triangular lattice system in the case of $N_1=13$, $N_2=15\approx(2/\sqrt{3})N_1=15.01110\ldots$ and $n_l=3$. Here we chose the nonzero elements in the upper triangle of the matrix $\widetilde{A}^{[n_l]}$ randomly from the positive region $(0.2\widetilde{\omega},1.8\widetilde{\omega})$ with a (nonzero) real constant $\widetilde{\omega}$, and took the arithmetic average over this randomness. As in the one-dimensional model including the long-range interactions, the Lyapunov spectrum is separated into the part changing smoothly and the part having a stepwise structure, which appears in a region of small positive Lyapunov exponents.

A remarkable point in the two-dimensional triangular lattice model, which the one-dimensional models do not have, is the wide steps of the Lyapunov spectrum, especially the step consisting of four points and the step consisting of eight points shown in Fig. 6. To give an explanation for these wide steps, we should notice that in this model we set up a periodic boundary condition for each of the three directions of the triangular lattice. This boundary condition for each of the three directions can cause degeneracies in the Lyapunov spectrum like in the one-dimensional cases. Besides, in this model each site of the triangular lattice has two independent degrees of freedom corresponding to the two dimensionality of the particles, which can also cause degeneracy in the Lyapunov spectrum due to condition (59). Therefore we can get the step consisting of four points corresponding to each direction of the triangular lattice. Moreover two of the three directions (namely, the two directions other than the horizontal direction in Fig. 5) have the same numbers of the sites in themselves, so they can cause a degeneracy in the Lyapunov spectrum. After all we get the step consisting of four points corresponding to the horizontal direction, and the step consisting of eight points corresponding to the other two directions. This explanation is partly justified by the fact that the Lyapunov spectrum in the case of $N_1 = N_2$ has the step consisting of 12 points in the Lyapunov spectrum, shown in the inset of Fig. 6, where we gave the averaged Lyapunov spectrum normalized by the maximum Lyapunov exponent in the case of $N_1 = N_2 = 14$ and $n_l = 3$. Here, except for the numbers N_1 and N_2 we used the same boundary condition and the same randomness for nonzero elements of the matrix $\widetilde{A}^{[n_l]}$ as in the case of $N_1 = 13$ and $N_2 = 15$ in Fig. 6. This Lyapunov spectrum is shifted by -2 in the j direction, so that the positions of the zero Lyapunov exponents coincide with the cases of $N_1 = 13$ and $N_2 = 15$. One may also notice that the parts changing smoothly in the Lyapunov spectrum are almost indistinguishable in both the cases in the inset of Fig. 6.

Like in the one-dimensional model, longer range interactions lead to a shorter region of stepwise structure in the Lyapunov spectrum in the two-dimensional triangular lattice model. On the other hand, a numerical simulation of the many-disk system showed that the region of the stepwise structure depends on the aspect ratio of the rectangular system [32]. This fact suggests that the long range interactions in the random matrix approach should depend on the aspect ratio of the rectangular system.

It should be emphasized that such wide step consisting of four points or eight points are actually found in the two-dimensional deterministic chaotic system consisting of many hard-core disks numerically [32]. Besides, we should also notice that in Fig. 6 (and Fig. 2) a warp of the Lyapunov spectrum appears in a region of large Lyapunov exponents, which is also a characteristic of the system consisting of many hard-core disks. However, there are some points for which the triangular lattice model in this section cannot give enough explanation when compared with the numerically observed features of the Lyapunov spectrum for the deterministic two-dimensional hard disk system. First, in the numerical simulation of the hard-disk system each particle can interact with almost any other particle in the long time limit. This should correspond to a big number of $n_l \approx N_1 N_2$ in the triangular lattice model, but if we adopt such large number for n_l then the stepwise structure disappears in the triangular lattice model. In this sense, the model in this section may correspond to the high density case, in which each particle mainly interacts with only a few particles surrounding it. Second, in the hard-disk system with a square shape the stepwise structure of the Lyapunov spectrum seems to be a repetition of the step consisting of four points and the step consisting of eight points. On the other hand, in the triangular lattice model such a repetition of steps cannot be guaranteed. As the third point, in the triangular lattice model in this section we adopt the boundary condition to make each of the three directions of the triangular lattice periodically, rather than the periodic boundary condition to make the up side (the left side) and the down side (the right side) of the square equivalently, which was adopted in the numerical simulation of the hard-disk system.

VII. CONCLUSION AND REMARKS

In this paper we have discussed the Lyapunov spectrum in many-particle systems described by a random matrix dynamics. We started from the many-particle Hamiltonian mechanics with no external force field, and introduced the Gaussian white random interactions between the particles. In such a system the dynamics of the tangent space is expressed by a Fokker-Planck equation, which leads to a direct connection between the positive (and zero) Lyapunov exponents and the time correlation of the matrix specifying the particle interactions. Using this formula, we calculated concretely the Lyapunov spectra in one- and two-dimensional models satisfying the total momentum conservation with periodic boundary conditions. These models show a stepwise structure of the Lyapunov spectrum in the region of small positive Lyapunov exponents, which is robust to a perturbation in matrix elements of the particle interaction matrix. The long range interactions between the particles lead to a clear separation into a part exhibiting stepwise structure and a part changing smoothly. The part of the Lyapunov spectrum containing stepwise structure is clearly distinguished by a wavelike structure in the eigenstates of the particle interaction matrix. In the two-dimensional model we got wider steps in the Lyapunov spectrum than in the one-dimensional models, especially the steps consisting of four points and the steps consisting of eight points. These wide steps in the Lyapunov spectrum have already been shown numerically in a deterministic chaotic system consisting of many hard-core disks.

One of the important simplifications in this random matrix approach is that in this approach we do not have to refer to the phase space dynamics in order to determine the tangent space dynamics anymore. In general, the matrix $R(t)$ appearing in Eq. (8) can depend on the phase space dynamics, so this kind of separation of the phase space dynamics and the tangent space dynamics cannot be allowed in deterministic chaotic systems.

As emphasized in the discussion of the two-dimensional model, a lattice picture is useful to make a model in the random matrix approach. Concerning this point it may be interesting to note that some works for the Lyapunov spectra for many-particle systems indicated a similarity between the solid state phenomena and the behavior of the Lyapunov spectra for many-particle systems in a fluid phase [16,40].

We have regarded the random matrix dynamics in this paper as an imitation of the deterministic chaotic dynamics, and reproduced some characteristics of chaotic systems, especially the stepwise structure of the Lyapunov spectra. However, we must not forget that there are some differences between the chaotic dynamics and the random matrix dynamics in this paper. For example, in the random matrix dynamics the movement of particles is not deterministic but stochastic, so that the zero Lyapunov exponents arising from the initial infinitesimal perturbation along the orbit in the deterministic chaos do not appear in the random matrix dynamics. (Note that we got only the zero Lyapunov exponents corresponding to the total momentum conservation in the models discussed in this paper.) We should also mention that in the random matrix approach there is an additional statistical average over the randomness of the interactions of particles, which does not exist in the deterministic dynamics. This causes vagueness in the definition of the Lyapunov exponents. In this paper we introduced the Lyapunov exponent as the time averaged *exponential rate of the randomness-averaged time evolution* of a neighboring trajectory. This definition allows us to get a simple connection between the Lyapunov exponents and the time correlation of the interaction matrix as shown in Eq. (26). However, this definition of the Lyapunov exponent is not proper to discuss the negative Lyapunov exponents and hence the pairing rule for the Lyapunov spectrum. On the other hand, we could adopt the definition of the Lyapunov exponent as the time average of *the randomness-averaged exponential rate of time evolution* of a neighboring trajectory. This definition should be proper to discuss the negative Lyapunov exponents. The comparison of these two definitions of the Lyapunov exponent is an unsettled problem.

One may regard the random matrix approach using the master equation in this paper as one of the analytical approaches to calculate the Lyapunov exponents. The other statistical and analytical approach for the Lyapunov exponents is the kinetic theoretical approach of Refs. [39,41]. In this approach the positive Lyapunov exponents are calculated using a Lorentz-Boltzmann equation, while the negative Lyapunov exponents are calculated using an ‘‘anti-Lorentz-Boltzmann equation’’ where the collision operator has the opposite sign to the ordinary Lorentz-Boltzmann equation. However, so far this kinetic theoretical approach can only provide the maximum Lyapunov exponent and the Kolmogorov-Sinai entropy for dilute gases. As another analytical approach to the Lyapunov exponents, one may also mention the geometric approach [42]. This approach can provide the largest Lyapunov exponent in terms of the average and fluctuations of the curvature of the configuration space.

To improve the random matrix approach used in this paper it is essential to know the statistical information for the interaction matrix $R(t)$. It has already been shown numerically that the time average of the matrix $R(t)$ is almost null in the system consisting of many hard-core disks [43]. This result justifies the condition (9) in the case of $n=1$. A similar investigation of the correlation of the matrix $R(t)$ in deterministic many-particle chaotic systems is one of the important future problems.

ACKNOWLEDGMENT

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APPENDIX A: MASTER EQUATION FOR THE TANGENT SPACE

In this appendix we derive the Fokker-Plank equation (13) for the tangent vector space. Using the Kramers-Moyal expansion the dynamics of the probability density $\rho(\delta\boldsymbol{\Gamma},t)$ is given by

$$\frac{\partial\rho(\delta\boldsymbol{\Gamma},t)}{\partial t}=\sum_{n=1}^{\infty}\sum_{j_1=1}^{2N}\sum_{j_2=1}^{2N}\cdots\sum_{j_n=1}^{2N}(-1)^n \times\frac{\partial^n \Xi^{(n)}_{j_1 j_2\cdots j_n}(\delta\boldsymbol{\Gamma},t)\rho(\delta\boldsymbol{\Gamma},t)}{\partial\delta\Gamma_{j_1}\,\partial\delta\Gamma_{j_2}\cdots\partial\delta\Gamma_{j_n}}, \tag{A1}$$

where $\Xi^{(n)}_{j_1 j_2\cdots j_n}(\delta\boldsymbol{\Gamma},t)$ is defined by

$$\Xi^{(n)}_{j_1 j_2\cdots j_n}(\delta\boldsymbol{\Gamma},t)\equiv\frac{1}{n!}\lim_{s\to 0}\frac{1}{s}\langle[\delta\Gamma_{j_1}(t+s)-\delta\Gamma_{j_1}(t)] \times[\delta\Gamma_{j_2}(t+s)-\delta\Gamma_{j_2}(t)]\cdots[\delta\Gamma_{j_n}(t+s) -\delta\Gamma_{j_n}(t)]\rangle|_{\delta\boldsymbol{\Gamma}(t)=\delta\boldsymbol{\Gamma}} \tag{A2}$$

and $\delta\Gamma_j(t)$ is the jth component of the tangent vector $\delta\boldsymbol{\Gamma}(t)$ [37].

It follows from Eqs. (2), (5), (6), and (8) that

$$\delta\boldsymbol{\Gamma}(t+s)-\delta\boldsymbol{\Gamma}(t)=\left\{\overleftarrow{T}\exp\left[J\int_t^{t+s}d\tau\;L(\tau)\right]-1\right\}\delta\boldsymbol{\Gamma}(t) =\sum_{n=1}^{\infty}\int_t^{t+s}d\tau_n\int_t^{\tau_n}d\tau_{n-1}\cdots\int_t^{\tau_2}d\tau_1 \times JL(\tau_n)JL(\tau_{n-1})\cdots JL(\tau_1)\delta\boldsymbol{\Gamma}(t), \tag{A3}$$

and

$$JL(\tau_n)JL(\tau_{n-1})\cdots JL(\tau_1) =\begin{cases}\begin{pmatrix}\underline{0} & I/m\\ R(\tau_1) & \underline{0}\end{pmatrix} & \text{for } n=1,\\ \begin{pmatrix}\Phi_l^{(1)} & \underline{0}\\ \underline{0} & \Phi_l^{(2)}\end{pmatrix} & \text{for } n=2l,\ \ l=1,2,\ldots,\\ \begin{pmatrix}\underline{0} & \Phi_l^{(2)}/m\\ m\Phi_{l+1}^{(1)} & \underline{0}\end{pmatrix} & \text{for } n=2l+1,\ \ l=1,2,\ldots.\end{cases} \tag{A4}$$

where $\Phi_l^{(j)}$, $j=1,2$ are defined by

$$\Phi_l^{(1)}\equiv R(\tau_{2l-1})R(\tau_{2l-3})\cdots R(\tau_1)/m^l, \tag{A5}$$

$$\Phi_l^{(2)} \equiv R(\tau_{2l})R(\tau_{2l-2})\cdots R(\tau_2)/m^l. \tag{A6}$$

By using Eqs. (9), (10), (A2), (A3), and (A4) we obtain

$$\begin{aligned}\boldsymbol{\Xi}^{(1)}(\delta\boldsymbol{\Gamma},t) &\equiv [\Xi_1^{(1)}(\delta\boldsymbol{\Gamma},t),\Xi_2^{(1)}(\delta\boldsymbol{\Gamma},t),\ldots,\Xi_{2N}^{(1)}(\delta\boldsymbol{\Gamma},t)]^T\\ &= \lim_{s\to 0}\frac{1}{s}\langle[\delta\boldsymbol{\Gamma}(t+s)-\delta\boldsymbol{\Gamma}(t)]\rangle|_{\delta\boldsymbol{\Gamma}(t)=\delta\boldsymbol{\Gamma}}\\ &= \lim_{s\to 0}\frac{1}{s}\int_t^{t+s} d\tau\langle JL(\tau)\rangle\delta\boldsymbol{\Gamma}\\ &= (\delta p_1,\delta p_2,\ldots,\delta p_N,0,0,\ldots,0)^T/m, \end{aligned}\tag{A7}$$

$$\begin{aligned}\Xi^{(2)}(\delta\boldsymbol{\Gamma},t) &\equiv [\Xi_{jk}^{(2)}(\delta\boldsymbol{\Gamma},t)] = \lim_{s\to 0}\frac{1}{2s}\langle[\delta\boldsymbol{\Gamma}(t+s)-\delta\boldsymbol{\Gamma}(t)]\\ &\quad\times[\delta\boldsymbol{\Gamma}(t+s)-\delta\boldsymbol{\Gamma}(t)]^T\rangle|_{\delta\boldsymbol{\Gamma}(t)=\delta\boldsymbol{\Gamma}}\\ &= \lim_{s\to 0}\frac{1}{2s}\int_t^{t+s}d\kappa\int_t^{t+s}d\tau\\ &\quad\times\langle JL(\kappa)\delta\boldsymbol{\Gamma}\,\delta\boldsymbol{\Gamma}^T[JL(\tau)]^T\rangle = \begin{pmatrix}\underline{0} & \underline{0}\\ \underline{0} & \Psi^{(2)}(\delta\mathbf{q})\end{pmatrix},\end{aligned}\tag{A8}$$

where $\Psi^{(2)}(\delta\mathbf{q})\equiv[\Psi_{jk}^{(2)}(\delta\mathbf{q})]$ is defined by

$$\Psi_{jk}^{(2)}(\delta\mathbf{q}) \equiv \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} D_{j\alpha\beta k}\delta q_\alpha\ \delta q_\beta. \tag{A9}$$

Here the only nonzero contributions come from the $n=1$ term of Eq. (A3). For general n, the number of δ functions from Eq. (10) must be only one less than the number of time integrals, to give a nonzero contribution. It is straightforward to show that this never happens for $n>1$. The terms including $\Xi_{j_1 j_2\cdots j_n}^{(n)}(\delta\boldsymbol{\Gamma},t)$, $n=3,4,\ldots$, in the right-hand side of Eq. (A1) are negligible because of the Gaussian white properties (9) and (10) of the random matrix $R(t)$. Using this fact and Eqs. (12), (A1), (A7), and (A8) we obtain the Fokker-Planck equation (13).

APPENDIX B: LYAPUNOV EXPONENTS IN THE RANDOM MATRIX APPROACH

In this appendix we derive the expression (26) for the Lyapunov exponents from the definition (25). Transforming Eqs. (18), (19), and (20) into the equations for the quantities $\tilde{\Upsilon}_{jk}^{(l)}(t)$ we obtain

$$\frac{d\tilde{\Upsilon}_{jk}^{(1)}(t)}{dt} = \frac{2}{m}\tilde{\Upsilon}_{jk}^{(2)}(t), \tag{B1}$$

$$\frac{d\tilde{\Upsilon}_{jk}^{(2)}(t)}{dt} = \frac{1}{m}\tilde{\Upsilon}_{jk}^{(3)}(t), \tag{B2}$$

$$\begin{aligned}\frac{d\tilde{\Upsilon}_{jk}^{(3)}(t)}{dt} &= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\sum_{\mu=1}^{N}\sum_{\nu=1}^{N} T_{j\alpha\beta k}D_{\alpha\mu\beta\nu}\Upsilon_{\mu\nu}^{(1)}(t)\\ &= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\sum_{\mu=1}^{N}\sum_{\nu=1}^{N} T_{j\alpha\beta k}D_{\alpha\mu\nu\beta}\\ &\quad\times\sum_{\alpha'=1}^{N}\sum_{\beta'=1}^{N}\sum_{\mu'=1}^{N}\sum_{\nu'=1}^{N} T_{\mu'\alpha'\beta'\nu'}\\ &\quad\times T_{\nu'\nu\mu\mu'}\Upsilon_{\alpha'\beta'}^{(1)}(t)\\ &= \Lambda_{jk}\tilde{\Upsilon}_{jk}^{(1)}(t),\end{aligned}\tag{B3}$$

where we used Eqs. (12), (21), (22), and (23). Equations (B1), (B2), and (B3) lead to

$$\frac{d^3\tilde{\Upsilon}_{jj}^{(1)}(t)}{dt^3} = \frac{2\Lambda_{jj}}{m^2}\tilde{\Upsilon}_{jj}^{(1)}(t), \tag{B4}$$

$$\frac{d^2\tilde{\Upsilon}_{jj}^{(1)}(t)}{dt^2} = \frac{2}{m^2}\tilde{\Upsilon}_{jj}^{(3)}(t), \tag{B5}$$

$$\frac{d\tilde{\Upsilon}_{jj}^{(1)}(t)}{dt} = \frac{2}{m}\tilde{\Upsilon}_{jj}^{(2)}(t). \tag{B6}$$

It is noted that Eq. (B4) is the differential equation only for the quantity $\tilde{\Upsilon}_{jj}^{(1)}(t)$.

If the quantity Λ_{jj} is zero, then the quantity $\tilde{\Upsilon}_{jj}^{(1)}(t)$ is a bilinear function of time t so that the Lyapunov exponent defined by Eq. (25) gives zero, namely, Eq. (26) is correct in this case. In the case of $\Lambda_{jj}\neq 0$, noting that the functions $\exp[(2\Lambda_{jj}/m^2)^{1/3}t\exp(2\pi ki/3)]$, $k=0,1,2$ are special solutions of Eq. (B4), we obtain the general solution

$$\begin{aligned}\tilde{\Upsilon}_{jj}^{(1)}(t) &= \Omega_j^{(1)}\ \exp(\mathcal{K}_j t) + \mathrm{Re}\bigg\{(\Omega_j^{(2)} - i\Omega_j^{(3)})\\ &\quad\times\exp\left[\mathcal{K}_j t\exp\left(\frac{2}{3}\pi i\right)\right]\bigg\}\\ &= \Omega_j^{(1)}\exp(\mathcal{K}_j t)\\ &\quad+\left[\Omega_j^{(2)}\cos\left(\frac{\sqrt{3}}{2}\mathcal{K}_j t\right)+\Omega_j^{(3)}\sin\left(\frac{\sqrt{3}}{2}\mathcal{K}_j t\right)\right]\\ &\quad\times\exp\left(-\frac{1}{2}\mathcal{K}_j t\right)\end{aligned}\tag{B7}$$

of Eq. (B4) for the real function $\tilde{\Upsilon}_{jk}^{(1)}(t)$ where $\mathcal{K}_j$ is defined by

$$\mathcal{K}_j \equiv \left(\frac{2\Lambda_{jj}}{m^2}\right)^{1/3}. \tag{B8}$$

Here $\Omega_j^{(k)}$, $k=1,2,3$ are real constants and are connected to the initial condition as

$$\Omega_j^{(1)}=\frac{1}{3}\left[\tilde{\Upsilon}_{jj}^{(1)}(0)+\frac{2\tilde{\Upsilon}_{jj}^{(2)}(0)}{m\mathcal{K}_j}+\frac{2\tilde{\Upsilon}_{jj}^{(3)}(0)}{(m\mathcal{K}_j)^2}\right], \tag{B9}$$

$$\Omega_j^{(2)}=\frac{2}{3}\left[\tilde{\Upsilon}_{jj}^{(1)}(0)-\frac{\tilde{\Upsilon}_{jj}^{(2)}(0)}{m\mathcal{K}_j}-\frac{\tilde{\Upsilon}_{jj}^{(3)}(0)}{(m\mathcal{K}_j)^2}\right], \tag{B10}$$

$$\Omega_j^{(3)}=\frac{2}{m\mathcal{K}_j\sqrt{3}}\left[\tilde{\Upsilon}_{jj}^{(2)}(0)-\frac{\tilde{\Upsilon}_{jj}^{(3)}(0)}{m\mathcal{K}_j}\right], \tag{B11}$$

by using Eqs. (B5), (B6), and (B7). By substituting Eq. (B7) into Eq. (25) we obtain

$$\lambda_j=\frac{\mathcal{K}_j}{2}=\left[\frac{\Lambda_{jj}}{(2m)^2}\right]^{1/3}, \tag{B12}$$

namely, Eq. (26).

Equations (B5) and (B6) imply that the quantities $\tilde{\Upsilon}_{jj}^{(2)}(t)$ and $\tilde{\Upsilon}_{jj}^{(3)}(t)$ are given by taking the first and second derivative of the quantity $\tilde{\Upsilon}_{jj}^{(1)}(t)$ with respect to the time t, respectively. This fact leads to the relation

$$\lim_{t\to\infty}\frac{1}{2t}\ln\frac{\tilde{\Upsilon}_{jj}^{(1)}(t)}{\tilde{\Upsilon}_{jj}^{(1)}(0)}=\lim_{t\to\infty}\frac{1}{2t}\ln\frac{\tilde{\Upsilon}_{jj}^{(2)}(t)}{\tilde{\Upsilon}_{jj}^{(2)}(0)} \tag{B13}$$

$$=\lim_{t\to\infty}\frac{1}{2t}\ln\frac{\tilde{\Upsilon}_{jj}^{(3)}(t)}{\tilde{\Upsilon}_{jj}^{(3)}(0)}. \tag{B14}$$

Namely, we get the same Lyapunov exponents as in Eq. (25) through the equations $\lim_{t\to+\infty}(2t)^{-1}\ln[\tilde{\Upsilon}_{jj}^{(k)}(t)/\tilde{\Upsilon}_{jj}^{(k)}(0)]$, $k=2,3$.

APPENDIX C: LYAPUNOV EXPONENTS FOR ONE-DIMENSIONAL MODELS WITH THE NEAREST-NEIGHBOR INTERACTIONS

In this appendix we give the derivation of Eqs. (38) and (39) from Eqs. (34) and (37). We also show Eqs. (41) and (42) in the case of $A=A^{(0)}+\varepsilon A^{(1)}$.

We define the discretized Fourier-transform matrix $F\equiv(F_{jk})$ by

$$F_{kl}\equiv\frac{1}{\sqrt{N}}\exp\left(-2\pi i\frac{kl}{N}\right), \tag{C1}$$

which is an unitary matrix: $F^\dagger F=FF^\dagger=I$ with the superscript † representing the Hermitian conjugate of the matrix.

First, it follows from Eqs. (37) and (C1) that

$$\begin{aligned}\sum_{\alpha=1}^{N}A_{j\alpha}^{(0)}F_{\alpha k}=&\frac{\omega}{\sqrt{N}}\{-2\exp(-2\pi ijk/N)\\&+\theta_{Nj}\theta_{j2}\exp[-2\pi i(j-1)k/N]\\&+\theta_{(N-1)j}\theta_{j1}\exp[-2\pi i(j+1)k/N]\\&+\theta_{Nj}\theta_{jN}\exp[-2\pi i(j-N+1)k/N]\\&+\theta_{1j}\theta_{j1}\exp[-2\pi i(j+N-1)k/N]\}\\=&a_k^{(0)}F_{jk},\end{aligned} \tag{C2}$$

where θ_{jk} is defined by

$$\theta_{jk}\equiv\begin{cases}1 & \text{in } j\geqslant k,\\ 0 & \text{in } j<k,\end{cases} \tag{C3}$$

and satisfies the relations $\theta_{jk}\theta_{kj}=\delta_{jk}$ and $\theta_{Nn}\theta_{nl}+\theta_{(l-1)n}\theta_{n1}=1$ for any integer $l\in\{2,3,\ldots,N\}$ and any integer $n\in\{1,2,\ldots,N\}$. Here the kth eigenvalue $a_k^{(0)}$ of the matrix $A^{(0)}$ is given by

$$a_k^{(0)}=-2\omega\left(1-\cos\frac{2\pi k}{N}\right). \tag{C4}$$

By using the formula (34) for the eigenvalues $a_j=a_j^{(0)}$ given by Eq. (C4) we obtain Eqs. (38). The eigenvalues of the matrix $A^{(0)}$ have degeneracies because of the relation $a_{N-j}^{(0)}=a_j^{(0)}$ in $j<N/2$, so we obtain Eq. (39).

Equation (C2) implies that the matrix $A^{(0)}$ is diagonalized by using the matrix F: $(F^\dagger A^{(0)}F)_{jk}=a_j^{(0)}\delta_{jk}$. The eigenvector $\mathbf{x}_j^{(0)}$ of the matrix $A^{(0)}$ corresponding to the eigenvalue $a_j^{(0)}$ is represented as

$$\mathbf{x}_j^{(0)}=(F_{1j},F_{2j},\ldots,F_{Nj})^T, \tag{C5}$$

so that we have the relation $A^{(0)}\mathbf{x}_n^{(0)}=a_j^{(0)}\mathbf{x}_n^{(0)}$. This set of eigenvectors satisfies the completeness condition and is normalized, namely, $\sum_{\alpha=1}^{N}\mathbf{x}_\alpha^{(0)}\mathbf{x}_\alpha^{(0)\dagger}=I$ and $\mathbf{x}_j^{(0)\dagger}\mathbf{x}_k^{(0)}=\delta_{jk}$.

Second, we expand the jth eigenvalue a_j of the matrix $A=A^{(0)}+\varepsilon A^{(1)}$ in the small parameter ε, namely, $a_j=a_j^{(0)}+\varepsilon a_j^{(1)}+\cdots$, and consider the first order correction $\varepsilon a_j^{(1)}$ by using the well-known perturbation theory for a degenerate system. We introduce the eigenstate $\mathbf{x}_j$ of the matrix A corresponding to the eigenvalue a_j, and expand it with the complete set $\{\mathbf{x}_l^{(0)}\}_l$ of the vectors,

$$\mathbf{x}_j=\sum_{\alpha=1}^{N}c_{j\alpha}\mathbf{x}_\alpha^{(0)}, \tag{C6}$$

with the constant $c_{j\alpha}\equiv\mathbf{x}_\alpha^{(0)\dagger}\mathbf{x}_j$. The relation $A\mathbf{x}_j=a_j\mathbf{x}_j$ is translated into the equation

$$\sum_{\alpha=1}^{N}[(a_k^{(0)}-a_j)\delta_{k\alpha}+\varepsilon W_{k\alpha}]c_{j\alpha}=0 \tag{C7}$$

for the coefficients $\{c_{jk}\}_{j,k}$ and the eigenvalue a_j with the quantity

$$W_{jk} \equiv \mathbf{x}_j^{(0)\dagger} A^{(1)} \mathbf{x}_k^{(0)}. \tag{C8}$$

We expand the coefficient c_{jk} by the small parameter ε, $c_{jk} = c_{jk}^{(0)} + \varepsilon c_{jk}^{(1)} + \cdots$. Now we calculate the first order corrections $\varepsilon a_j^{(1)}$ and $\varepsilon a_{N-j}^{(1)}$ of the eigenvalues a_j and a_{N-j}, respectively, which have a degeneracy in the zeroth order of the parameter ε. For this purpose, instead of the coefficients $\{c_{jk}\}_{j,k}$ appearing in Eq. (C7) it is enough to consider the zeroth order coefficients $\{c_{jk}^{(0)}\}_{j,k}$, which are zero except for $c_{jj}^{(0)}$, $c_{j(N-j)}^{(0)}$, $c_{(N-j)j}^{(0)}$, and $c_{(N-j)(N-j)}^{(0)}$. This leads to the equation

$$\mathrm{Det}\begin{pmatrix} -a^{(1)} + W_{jj} & W_{j(N-j)} \\ W_{(N-j)j} & -a^{(1)} + W_{(N-j)(N-j)} \end{pmatrix} = 0 \tag{C9}$$

for $a^{(1)}$, whose solutions give the eigenvalues $a_j^{(1)}$ and $a_{N-j}^{(1)}$. We can solve Eq. (C9) and obtain

$$a^{(1)} = \frac{W_{jj} + W_{(N-j)(N-j)}}{2} \pm \sqrt{\left(\frac{W_{jj} - W_{(N-j)(N-j)}}{2}\right)^2 + |W_{j(N-j)}|^2}, \tag{C10}$$

noting the relation $W_{kj} = W_{jk}^*$ with the superscript * representing the complex conjugate of the complex number. The eigenvalue $a_N^{(0)}$ has no degeneracy so we simply have $a_N^{(1)} = W_{NN}$. If the number N is even, then the eigenvalue $a_{N/2}^{(0)}$ has no degeneracy, so we also have $a_{N/2}^{(1)} = W_{(N/2)(N/2)}$ in this case.

By using Eqs. (40), (C1), (C5), and (C8) we obtain

$$W_{kl} = -[1 - \exp(-2\pi i k/N)] \times [1 - \exp(2\pi i l/N)] N^{-1} \times \sum_{\alpha=1}^{N} \chi_\alpha \exp[2\pi i \alpha (k - l)/N]. \tag{C11}$$

Especially we derive

$$W_{kk} = -|1 - \exp(-2\pi i k/N)|^2 \tilde{\chi}_0 = \frac{\tilde{\chi}_0}{\omega} a_k^{(0)} = W_{(N-k)(N-k)}, \tag{C12}$$

$$W_{k(N-k)} = -[1 - \exp(-2\pi i k/N)]^2 \tilde{\chi}_k \tag{C13}$$

from Eqs. (43), (C4), and (C11). By substituting Eqs. (C12) and (C13) into Eq. (C10) we obtain

$$a_j^{(1)} = \frac{a_j^{(0)}}{\omega} (\tilde{\chi}_0 + |\tilde{\chi}_j|), \tag{C14}$$

$$a_{N-j}^{(1)} = \frac{a_{N-j}^{(0)}}{\omega} (\tilde{\chi}_0 - |\tilde{\chi}_j|) \tag{C15}$$

in the case of $j < N/2$, where we used the relation $|W_{j(N-j)}| = |\tilde{\chi}_j a_j^{(0)}/\omega|$ and numbered so that we obtain $(a_j^{(1)} - a_{N-j}^{(1)})\omega/a_j^{(0)} \geqslant 0$. By using Eq. (C12) we also obtain $a_{N/2}^{(1)} = \tilde{\chi}_0 a_{N/2}^{(0)}/\omega$ in the case of the number N to be even, and $a_N^{(1)} = 0$ for the first order correction to the nondegenerate eigenvalues of the matrix $A^{(0)}$. By applying the formula (34) to the case of $a_j = a_j^{(0)} + \varepsilon a_j^{(1)} + O(\varepsilon^2)$ with the quantity $a_j^{(1)}$ given by Eq. (C14) or (C15), we obtain Eqs. (41) and (42).

APPENDIX D: LYAPUNOV EXPONENTS FOR ONE-DIMENSIONAL MODELS WITH LONG RANGE INTERACTIONS OF A FIXED STRENGTH

In this appendix we calculate the Lyapunov spectra for the systems with long range interactions of a fixed strength. First we consider the case described by the matrix $A = \overline{A}^{[n_l]} \equiv (\overline{A}_{jk}^{[n_l]})$ defined by

$$\overline{A}_{jk}^{[n_l]} = -2n_l \overline{\omega} \delta_{jk} + \overline{\omega} \sum_{l=1}^{n_l} [\delta_{j(k+l)} + \delta_{k(j+l)} + \delta_{j(k+N-l)} + \delta_{k(j+N-l)}], \tag{D1}$$

with $n_l < N/2$ and a (nonzero) real constant $\overline{\omega}$. It should be noted that the matrix $A^{[n_l]}$ given by Eq. (46) is attributed into this matrix $\overline{A}^{[n_l]}$ in the case of $\sigma_j^{[l]} = \overline{\omega}$. The matrix $\overline{A}^{[n_l]}$ is diagonalized by the matrix F defined by Eq. (C1), and we obtain the relation $(F^\dagger \overline{A}^{[n_l]} F)_{jk} = \overline{a}_k^{[n_l]} \delta_{jk}$ with the eigenvalue $\overline{a}_j^{[n_l]} \equiv -2\overline{\omega}[n_l - \sum_{l=1}^{n_l} \cos(2\pi j l/N)]$. By applying the formula (34) to the eigenvalue $a_j = \overline{a}_j^{[n_l]}$ we obtain the Lyapunov exponent

$$\lambda_j = \left|\frac{\overline{\omega}}{m}\right|^{2/3} \left[n_l - \sum_{l=1}^{n_l} \cos\left(\frac{2\pi j l}{N}\right)\right]^{2/3}. \tag{D2}$$

The Lyapunov exponents given by Eq. (D2) satisfies the relation $\lambda_j = \lambda_{N-j}$ in $j < N/2$, so the Lyapunov spectrum of this system has a stepwise structure.

Second we consider the case that each particle interacts with all the other particles with the same strength. This is described by the matrix $A = \overline{A}^{[N/2]} \equiv (\overline{A}_{jk}^{[N/2]})$, which is defined by

$$\overline{A}_{jk}^{[N/2]} \equiv 2\overline{\omega}(1 - N\delta_{jk}), \tag{D3}$$

namely, the matrix whose off-diagonal elements are nonzero and equal each other. The matrix (D3) is diagonalized as

$$(F^\dagger \overline{A}^{[N/2]} F)_{jk} = -2N\overline{\omega}(1 - \delta_{jN})\delta_{jk} \tag{D4}$$

by using the matrix F defined by Eq. (C1), so the eigenvalues of the matrix $\overline{A}^{[N/2]}$ are $-2N\overline{\omega}$ and 0. By substituting the eigenvalues of the matrix $A=A^{[N/2]}$ in the formula (34) we obtain the Lyapunov spectrum as

$$\lambda^{[j]}=\begin{cases} \left|\dfrac{N\overline{\omega}}{m}\right|^{2/3} & \text{in } j=1,2,\ldots,N-1, \\ 0 & \text{in } j=N, \end{cases} \tag{D5}$$

namely, the shape of positive Lyapunov spectrum in this system is just in a straight horizontal line. It should be noted that the quantity $\overline{\omega}$ may depend on the number N of the particles in general. Therefore the consideration in this appendix is not enough to discuss the particle number dependence of the maximum Lyapunov exponent.

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